Grade 7/8 Math Circles
November 21/22/23/24, 2022
Complex Numbers

Motivation

Definition

- The set of all real numbers, $\mathbb{R}$, contains all numbers on the (real) number line.

Stop and Think

1. Think about the equation $x^2 = 4$. What is a possible real x value?
2. Think about the equation $x^2 = 2$. What is a possible real x value?
3. Think about the equation $x^2 = -1$. What is a possible real x value?

Stop and Think Explained

1. Consider $x = \pm 2$ (this means $x = 2$ or $x = -2$, and it is read "plus or minus two").
2. We use the $\sqrt{\quad}$ (square root) sign to indicate that when you multiply that number by itself, you will get the number inside of the square root sign. For instance, we have $\sqrt{2} \times \sqrt{2} = 2$. So, the square roots of 2 will be $\pm\sqrt{2}$.
3. This is a trick question. There is no such possible real x value. Since two positive real numbers multiplied together is positive, any positive real number multiplied by itself is positive. For example, 1 is positive, and $1 \times 1 = 1$, which is positive. Similarly, since two negative real numbers multiplied together is positive, any negative real number multiplied by itself is also positive. For example, -1 is negative, and $(-1) \times (-1) = 1$, which is positive. Hence, you cannot get a negative real number, -1, by multiplying a real number by itself.

Mathematicians wondered, is there a number that can be squared (multiplied by itself) to get a negative real number? Here is the answer that they came up with:

Definition and Rules

- The **imaginary unit**, denoted i, is a number with the property that $i^2 = -1$, that is $i = \sqrt{-1}$.
- If $a > 0$, $\sqrt{-a}$ can be written as

$$\sqrt{-a} = \sqrt{(-1)a} = \sqrt{-1}\sqrt{a} = \sqrt{a}i.$$

  Since $i^2 = -1$, this can be verified by noticing that $(\sqrt{a}i)^2 = ai^2 = -a$. So $\sqrt{a}i$ is a square root of $-a$. Similarly, you can also verify that $-\sqrt{a}i$ is a square root of $-a$.
- The rule $\sqrt{a} \times \sqrt{b} = \sqrt{a \times b}$ is not valid when both a and b are negative real numbers.

For the third point in the Definition and Rules, it is very important that not both a and b are negative real numbers. If we remove this requirement, we create a HUGE problem. Consider the following:

$$-1 = i \times i = \sqrt{-1} \times \sqrt{-1} = \sqrt{(-1) \times (-1)} = \sqrt{1} = 1,$$

which is INCORRECT ($-1 = 1$ should never be the case...)

Example 1

Find the equivalent expression to the following using i:

1. $\sqrt{-9} = \sqrt{9}i = 3i.$
2. $\sqrt{-5} = \sqrt{5}i.$
3. $\sqrt{-72} = \sqrt{72}i = \sqrt{6^2 \times 2}i = \sqrt{6^2} \times \sqrt{2} \times i = 6\sqrt{2}i.$
4. $\sqrt{-\frac{9}{4}} = \sqrt{\frac{9}{4}}i = \sqrt{\frac{3^2}{2^2}}i = \frac{3}{2}i.$

Exercise 1

Find the equivalent expression to the following using i:

1. $\sqrt{-16}$
2. $\sqrt{-7}$

3. $\sqrt{-75}$
4. $\sqrt{-\frac{1}{4}}$

Exercise 1 Solution

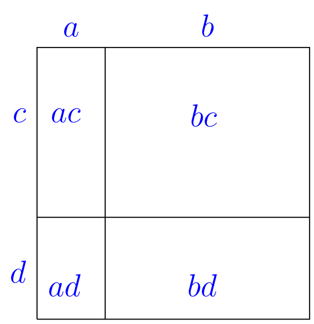
1. $4i$
2. $\sqrt{7}i$
3. $5\sqrt{3}i$
4. $\frac{1}{2}i$

The FOIL Method

We use the FOIL (First, Outside, Inside, Last) method to multiply two binomials (sum of two terms) together. If you are multiplying $(a+b)$ and $(c+d)$ together, you want to do the following:

1. First: multiply the first terms (a and c) together to obtain $a \times c = ac$.
2. Outside: multiply the outside terms (a and d) together to obtain $a \times d = ad$.
3. Inside: multiply the inside terms (b and c) together to obtain $b \times c = bc$.
4. Last: multiply the last terms (b and d) together to obtain $b \times d = bd$.
5. Take the sum of Steps 1-4 to obtain $ac + ad + bc + bd$.

To visualize, think about finding the area of a rectangle with the length $a + b$ and the width $c + d$.



To summarize, $(a+b) \times (c+d) = ac + ad + bc + bd$.

Definition of Complex Numbers

We can combine real numbers with the imaginary unit i to define a new type of numbers.

Definition

- A **complex number** is a number that can be expressed in the form $a + bi$ where a and b are real numbers and i is the imaginary unit. The real number a is called the **real part** of $a + bi$, and the real number b is called the **imaginary part** of $a + bi$.
- The complex numbers $a + bi$ and $c + di$ are **equal** (written $a + bi = c + di$), if $a = c$ and $b = d$.

Notice that if $b = 0$ in $a + bi$, then $a + bi = a$, which is a real number. Thus, all real numbers are also complex numbers.

Exercise 2

Find the real part and the imaginary part of the following complex numbers:

1. $2 + 3i$
2. $7i + \frac{1}{2}$
3. 17
4. $-i$

Exercise 2 Solution

1. The real part is 2 and the imaginary part is 3.
2. The real part is $\frac{1}{2}$ and the imaginary part is 7.
3. The real part is 17 and the imaginary part is 0.
4. The real part is 0 and the imaginary part is -1.

Operations on Complex Numbers

Definition

Let $a + bi$ and $c + di$ be complex numbers.

- We define **addition** by

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

- We define **subtraction** by

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

- We define **multiplication** by

$$(a + bi) \times (c + di) = (ac - bd) + (ad + bc)i$$

- We define **division** by

$$\frac{a + bi}{c + di} = \frac{(a + bi) \times (c - di)}{(c + di) \times (c - di)} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i \qquad (\text{If, } c + di \neq 0)$$

Notice that, by using the FOIL method, we can derive the definition of multiplication of complex numbers:

$$\begin{aligned}(a + bi) \times (c + di) &= ac + adi + bci + bdi^2 \\ &= ac + (ad + bc)i - bd \\ &= (ac - bd) + (ad + bc)i\end{aligned}$$

Example 2

Find the following complex numbers:

1. $(2 - 3i) + (-1 + 6i) = (2 + (-1)) + ((-3) + 6)i = 1 + 3i$

2. $(-7-2i)-(2+3i) = (-7-2)+(-2-3)i = -9-5i$
3. $(-1+3i)\times(2-3i) = ((-1)\times 2 - 3\times(-3)) + (-1\times(-3)+2\times 3)i = 7+9i$
4. $\dfrac{1-3i}{-2+i} = \dfrac{1\times(-2)+(-3)\times 1}{4+1} + \dfrac{(-3)\times(-2)-1\times 1}{4+1}i = \dfrac{-5}{5}+\dfrac{5}{5}i = -1+i$

Exercise 3

Find the following complex numbers:

1. $(3+2i)+(4-6i)$
2. $(-3+i)-(-2+7i)$
3. $(-2+i)\times(3-i)$
4. $\dfrac{-1+i}{3+2i}$

Exercise 3 Solution

1. $7-4i$
2. $-1-6i$
3. $-5+5i$
4. $\dfrac{-1+5i}{13}$

Let's take a closer look at the division of complex numbers. What is really going on when we are multiplying $(c+di)$ by $(c-di)$?

Definition

- The complex **conjugate** of a complex number $z = a+bi$, denoted $\overline{z} = \overline{a+bi}$, is the complex number
$$\overline{z} = a - bi.$$
- $z\times\overline{z} = (a+bi)(a-bi) = a^2+b^2$ is a real number. (Use the FOIL method!)
- If z is purely real, then $\overline{z} = z$.

- If z is purely imaginary, then $\overline{z} = -z$.

Notice when we calculate $\dfrac{a+bi}{c+di}$, the division with complex numbers, we multiply both the numerator and the denominator by the complex conjugate of $c+di$, so it results in a simpler expression. The value of the division remains the same because we are multiplying it by $1 = \dfrac{c-di}{c-di}$.

Properties

Let z_1 and z_2 be complex numbers.

- $\overline{(\overline{z_1})} = z_1$
- $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
- $\overline{z_1 \times z_2} = \overline{z_1} \times \overline{z_2}$
- $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}} \qquad (\text{If, } z_2 \neq 0)$

Powers of i

We can get the following powers of i:

$$\begin{aligned} i &= i \\ i^2 &= -1 \\ i^3 &= -i \\ i^4 &= 1 \\ i^5 &= i \\ i^6 &= -1 \\ i^7 &= -i \\ i^8 &= 1 \\ &\vdots \end{aligned}$$

Notice that $i, -1, -i, 1$ repeat in that order. So in general, if we have a natural number n, then

$$\begin{aligned} i^{4n} &= 1 \\ i^{4n+1} &= i \\ i^{4n+2} &= i^2 = -1 \\ i^{4n+3} &= i^3 = -i \end{aligned}$$

Example 3

1. $i^{10} = i^{4\times2+2} = -1$
2. $i^{103} = i^{4\times25+3} = -i$
3. $i^{97} = i^{4\times24+1} = i$
4. $i^{400} = i^{4\times100} = 1$

Exercise 4

Calculate the following powers of i:

1. i^{1003}
2. i^{16}
3. i^{90}
4. i^{45}

Exercise 4 Solution

1. $i^{1003} = i^{4\times250+3} = -i$
2. $i^{16} = i^{4\times4} = 1$
3. $i^{90} = i^{4\times22+2} = -1$
4. $i^{45} = i^{4\times11+1} = i$

Example 4

1. Using the FOIL method, we have $(1+i)^2 = 1 + 2i + i^2 = 2i$, then

$$(1+i)^6 = (1+i)^2 \times (1+i)^2 \times (1+i)^2 = [(1+i)^2]^3 = [2i]^3 = 8i^3 = -8i.$$

2. Using the FOIL method, we have $(1-i)^2 = 1 - 2i + i^2 = -2i$, then

$$(1-i)^{10} = [(1-i)^2]^5 = [-2i]^5 = -32i^5 = -32i.$$

The Quadratic Formula

Definition

- A **quadratic equation** is any equation that can be rearranged into the form:

$$ax^2 + bx + c = 0$$

where x represents an unknown, and a, b, and c represent known real numbers, where $a \neq 0$. The values of x which satisfy the equation are called **solutions** (or **zeroes**) of the equation.

Fortunately, we have a mathematical formula to find the solutions of quadratic equations.

Formula

- The **quadratic formula**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

gives the solutions of the quadratic equation $ax^2 + bx + c = 0$ in terms of a, b, and c.

The expression, $b^2 - 4ac$, underneath the square root sign is called the **discriminant** of the quadratic equation, and it is often denoted as D. The quadratic equation $ax^2 + bx + c = 0$ has different kinds/numbers of solutions based on the value of D.

1. If $D > 0$, then the equation has 2 distinct real solutions.
2. If $D = 0$, then the equation has 1 real solution, $-\frac{b}{2a}$.
3. If $D < 0$, then the equation has 2 distinct complex solutions which are complex conjugates of each other.

Example 5

Find the solutions of the quadratic equation $x^2 + 7x + 10 = 0$.

We use the quadratic formula. We have $a = 1, b = 7$, and $c = 10$. By substituting these values into the quadratic formula, we get

$$\begin{aligned} x &= \frac{-7 \pm \sqrt{7^2 - 4(1)(10)}}{2(1)} \\ &= \frac{-7 \pm \sqrt{9}}{2} \\ &= \frac{-7 \pm 3}{2} \\ &= \frac{-7+3}{2}, \frac{-7-3}{2} \\ \therefore x &= -2, -5 \end{aligned}$$

(The symbol $\therefore$ means "therefore".)

Exercise 5

Find the solutions of the following quadratic equations:

1. $x^2 + 6x + 9 = 0$
2. $2x^2 - 3x + 1 = 0$
3. $x^2 + 2x + 4 = 0$
4. $x^2 + 1 = 0$

Exercise 5 Solution

1. Using the quadratic formula, we get

$$\begin{aligned} x &= \frac{-6 \pm \sqrt{6^2 - 4(1)(9)}}{2(1)} \\ &= \frac{-6 \pm 0}{2} \\ \therefore x &= -3 \end{aligned}$$

2. Using the quadratic formula, we get

$$\begin{aligned} x &= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(1)}}{2(2)} \\ &= \frac{3 \pm \sqrt{1}}{4} \\ &= \frac{3 \pm 1}{4} \\ &= \frac{3+1}{4}, \frac{3-1}{4} \\ \therefore x &= 1, \frac{1}{2} \end{aligned}$$

3. Using the quadratic formula, we get

$$\begin{aligned} x &= \frac{-2 \pm \sqrt{2^2 - 4(1)(4)}}{2(1)} \\ &= \frac{-2 \pm \sqrt{-12}}{2} \\ &= \frac{-2 \pm 2\sqrt{3}i}{2} \\ &= -1 \pm \sqrt{3}i \\ \therefore x &= -1 + \sqrt{3}i, -1 - \sqrt{3}i \end{aligned}$$

4. Using the quadratic formula, we get

$$\begin{aligned} x &= \frac{0 \pm \sqrt{0^2 - 4(1)(1)}}{2(1)} \\ &= \pm\frac{\sqrt{-4}}{2} \\ &= \pm\frac{2i}{2} \\ &= \pm i \\ \therefore x &= i, -i \end{aligned}$$

However, we did not need to use the quadratic formula! By rearranging the equation, we get $x^2 = -1$ and by the definition of i, we already know that the solutions are $\pm i$.

Notice that not every quadratic equation has real solution(s). In case of $D < 0$, you will come across people saying "no solution", but that actually means "no real solution" because you can ALWAYS find complex solution(s) of every quadratic equation by using the quadratic formula.

Roots of One

Definition

- $\boldsymbol{n}^{\text{th}}$ **Roots of one** (or $\boldsymbol{n}^{\text{th}}$ **roots of unity**) are the solutions of the equation

$$x^n = 1$$

where n is a natural number.

Notice the following:

- 1^{st} root of one is the solution of the equation $x = 1$. So, the first root of one is 1.
- 2^{nd} roots of one are the solutions of the equation $x^2 = 1$. So, the second roots of one are 1 and -1. You can verify this by using the quadratic formula on $x^2 - 1 = 0$.
- 4^{th} roots of one are the solutions of the equation $x^4 = 1$. So, the fourth roots of one are $1, -1, i$, and $-i$. You can verify this by observing that $1^4 = (-1)^4 = i^4 = (-i)^4 = 1$.

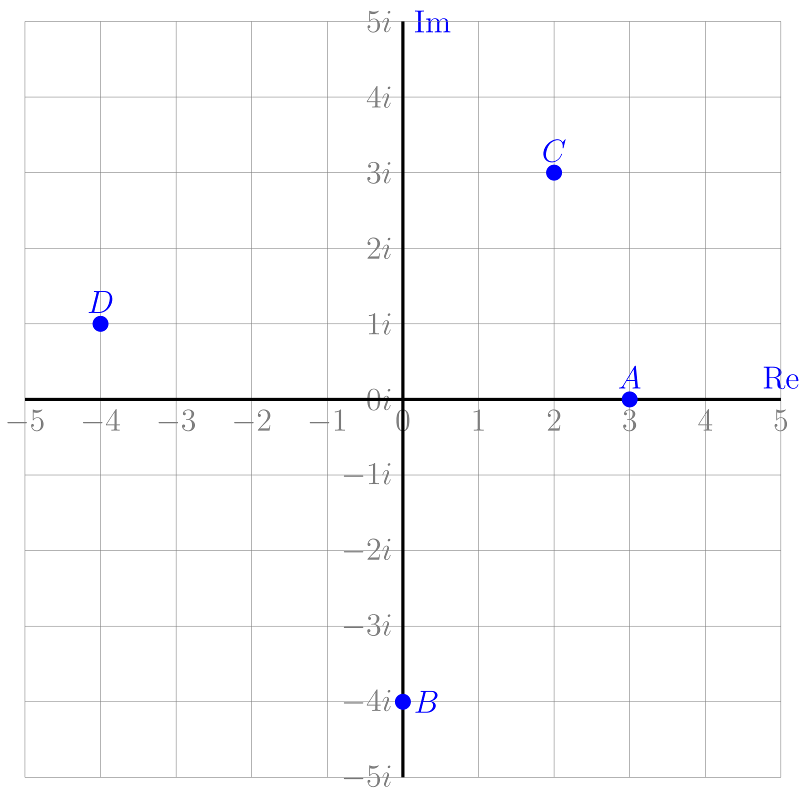
There is a way to plot complex numbers in the coordinate plane, using the real part as an x-coordinate and the imaginary part as a y-coordinate. As examples, we will plot the following points (complex numbers):

A. 3

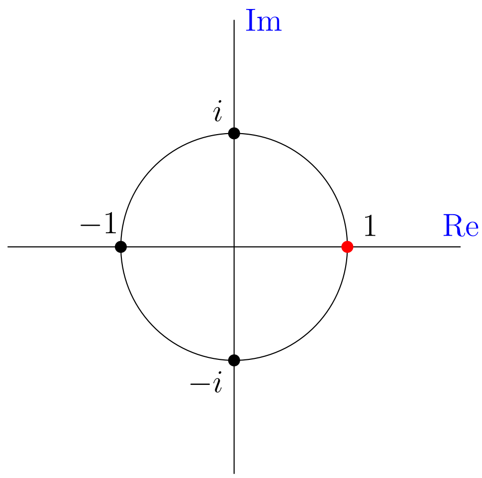
B. $-4i$

C. $2+3i$

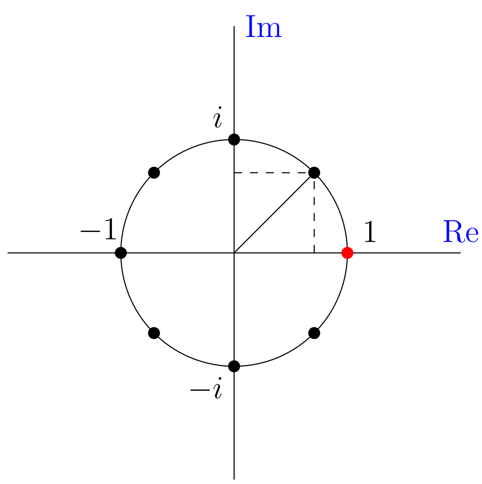
D. $-4+i$



When we plot roots of one in the same way, they all end up lying on the unit circle, the set of points of distance 1 from the origin. You can see all 4^{th} roots of one on the unit circle in the diagram below:



Geometrically, you can get all n^{th} roots of one by placing a point at 1 on the complex unit circle and equally spacing out $n-1$ points on the circle around 1. Here is a diagram showing all the 8^{th} roots of one:

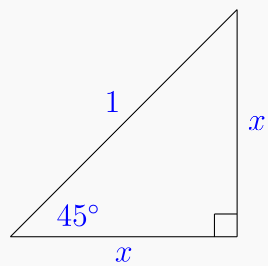


Stop and Think

Since $(\sqrt{i})^8 = [(\sqrt{i})^2]^4 = i^4 = 1$, an 8^{th} root of one is $\sqrt{i}$. Can you find an 8^{th} root of one other than $\pm 1, \pm i$ in the form of $a + bi$ where a and b are real numbers?
(Hint: Use the unit circle and the Pythagorean Theorem!)

Stop and Think Explained

Since the point between 1 and i is equally distant from 1 and i, we know that the radius of the unit circle passing through that point creates a $45°$ angle with the real number line (axis). By drawing a perpendicular line segment from that point to the real number line (axis), we get the following triangle:

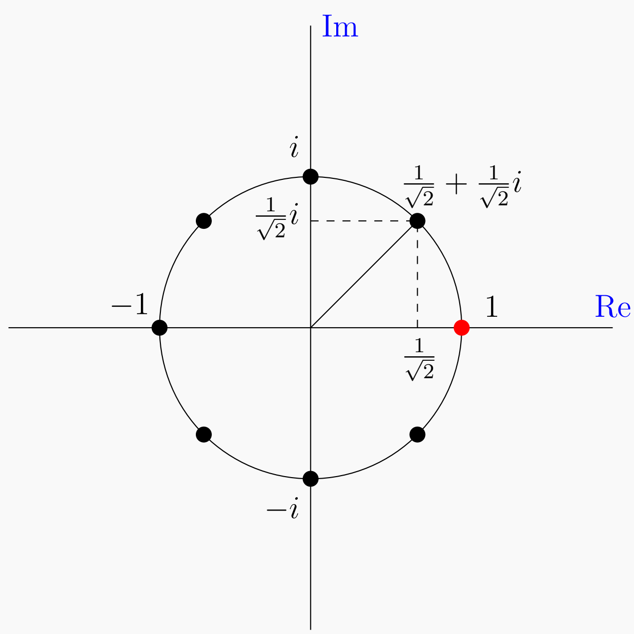


Using the Pythagorean Theorem, we have

$$1^2 = x^2 + x^2 = 2x^2.$$

Since x is the length, $x > 0$. Solving the above equation, we get $x = \dfrac{1}{\sqrt{2}}$.

Let's mark this information on our diagram:



So, $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ is an 8^{th} root of one. Other 8^{th} roots of one include $\pm 1, \pm i, \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i, -\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i$.